Elastic ep scattering and higher radiative corrections
Part I

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Abstract

Higher order correction in the ep scattering will be discussed. In particular the second order Born correction to the electron scattering off Coulomb potential will be derived. Various possible shapes of the charged distribution inside the proton will be discussed. Then I will evaluate the "box" contribution to the ep scattering, assuming that the intermediate state is given by the virtual nucleon. The theoretical results will be compared with the model independent prediction of the higher order contribution obtained with the neural networks. Eventually I will show the prediction of proton radius.

Plan of the Seminar

- Elementary Introduction
  - Electromagnetic Form Factors
  - The Proton Size
- Motivation;
  - Disagreement between "Rosenbluth" and "polarization transfer" data $\rightarrow$ TPE physics;
  - The problem of the proton size?
- Electron scattering off Coulomb potential
  - Point-like Coulomb potential;
  - Yukawa, and exponential charge distributions, what about others?
- Neural Network Approach
- Box Diagrams

- *electron-proton interaction: the problem as old as the quantum mechanics and particle physics*;
- *electron-A scattering: the most powerful tool for the investigation of the internal structure matter and fundamental forces*;
- ***The experimental data seem to be more precise than theoretical prediction;***

Before Quantum Electrodynamics

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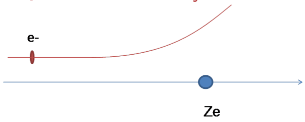


- Rutherford formula, E. Rutherford, Phil. Mag., **21**, 669 (1911):

$$\frac{d\sigma_{Ruth.}}{d\Omega} = \frac{Z^2 e^4}{16\pi^2 m^2 v_\infty^4} \frac{1}{\sin^4 \frac{\theta}{2}} = \frac{Z^2\alpha^2}{4E^2} \frac{1}{\sin^4 \frac{\theta}{2}} \tag{1}$$

- Schoedinger equation, solved in CMF (see: Schiff, *Mechanika Kwantowa*):

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{e}{r} - E \right] u(\mathbf{r}) = 0 \tag{2}$$

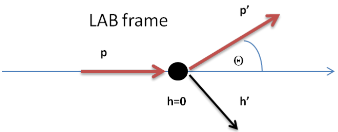
Hypergeometric functions, solution is given by distorted waves

$$e^{-i\mathbf{p}\cdot\mathbf{r} + i(\alpha/\beta)\ln(2pr\sin^2(\theta/2))} \tag{3}$$

- **Infinite range of the Coulomb Potential!**

- **Dirac Equation:** Dirac, Proc. R. Soc. Lond. A**117**, 610 (1928), *ibid* Proc. R. Soc. Lond. A**118**, 351 (1928)
- **Solution of the Dirac equation in terms of partial waves:** Darwin, Proc. R. Soc. Lond. A**118**, 654 (1928)
- **Dirac equation with Coulomb field, atomic collision, analytic solution:** Mott Proc. R. Soc. Lond. A**124**, 425 (1929)
- **Numerical expansion of the Mott solution:** McKinley, Feshbach, Phys. Rev. **74** (1948) 1759.

Elementary Introduction



- Elastic ep scattering:
$$e(p) + p(h) \Rightarrow e(p') + p(h') \tag{4}$$
- $p^\mu = (E, \mathbf{p})$, $p'^\mu = (E', \mathbf{p}')$, for $m_e \approx 0$, $|\mathbf{p}| = E$, $\mathbf{p}' = E'$, M denotes the proton mass

$$\mathbf{p} \cdot \mathbf{p}' = |\mathbf{p}||\mathbf{p}'| \cos\theta \tag{5}$$
$$q^\mu = p^\mu - p'^\mu = h'^\mu - h^\mu = (\nu, \mathbf{q}), \quad Q^2 \equiv -q^2 = 4EE' \sin^2\frac{\theta}{2} \tag{6}$$
$$\tau = \frac{Q^2}{4M^2} \tag{7}$$
$$\epsilon = \left(1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right)^{-1} \tag{8}$$

Point-like Proton $s = 1/2$

$$\mathcal{L}_{int}(x) = -e\mathcal{J}^{\mu}_{lep}(x)A_{\mu}(x) + e\mathcal{J}^{\mu}_{had}(x)A_{\mu}(x) = -\mathcal{H}_{int}(x) \quad (9)$$

$$\mathcal{J}^{\mu}_{lep} = \overline{\psi}_e(x)\gamma^{\mu}\psi_e(x), \quad \mathcal{J}^{\mu}_{had} = \overline{\psi}_p(x)\gamma^{\mu}\psi_p(x), \quad (10)$$

$$T_{fi} = \lim_{T\to 1-i\epsilon}\langle \mathbf{h}', \mathbf{p}' \mid T\left(\exp\left\{-i\int d^3x\int_{-T}^{T} dt\mathcal{H}_I(x)\right\}\right) \mid \mathbf{h}, \mathbf{p}\rangle_{conn.ampu.}$$

$$= T^{(0)}_{fi} + T^{(1)}_{fi} + T^{(2)}_{fi} + \ldots \quad (11)$$

$$T^{(k)}_{fi} = (-i)^k\langle \mathbf{h}', \mathbf{p}' \mid T\left\{\frac{1}{k!}\prod_{j=1}^{k}\int d^4x_j\mathcal{H}_{int}(x_j)\right\} \mid \mathbf{h}, \mathbf{p}\rangle_{conn.ampu.}$$

$$T^{(k)}_{fi} \sim (e^2)^k \quad (12)$$

$$T_{fi} = i(2\pi)^4\delta^{(4)}\left(p + h - p' - h'\right)\mathcal{M}_{fi} \quad (13)$$

$$d\sigma \sim |\mathcal{M}_{fi}|^2 \quad (14)$$

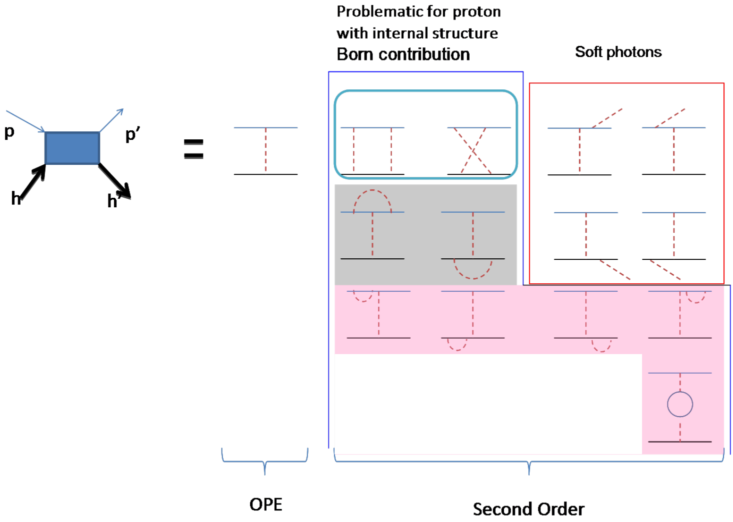


Figure: Infrared Divergent (IR) contribution!
$\sigma_{Exp} = \sigma_{Born+vertex+self-energies} + \sigma_{soft-photons}(\Delta E)$, ΔE energy resolution of the experimental apparatus.

Point-like Proton $s = 1/2$

- Self-Energy correction, renormalized photon and electron propagators;
- Vertex correction;
- Born contribution;
- L. W. Mo and Y. S. Tsai, *Radiative Corrections To Elastic And Inelastic E P And Mu P Scattering*, Rev. Mod. Phys. **41** (1969) 205.

Point-like Proton – OPE (One Photon Exchange)

$$J^{\mu}_{lep}(x) = \langle p', s' \mid \mathcal{J}^{\mu}_{lep}(x) \mid p, s\rangle, \quad J^{\mu}_{had}(x) = \langle h', r' \mid \mathcal{J}_{had}(x) \mid h, r\rangle \tag{15}$$

For point like particle

$$J^{\mu}_{lep}(x) = \overline{u}_{electron}(p', s')\gamma^{\mu} u_{electron}(p, s) e^{-ix\cdot(k'-k)}, \tag{16}$$

$$J^{\mu}_{had}(x) = \overline{u}_{proton}(h', r')\gamma^{\mu} u_{proton}(h, r) e^{-ix\cdot(h'-h)} \tag{17}$$

In the OPE (first Born) approximation:

$$i\mathcal{M}^{(1)}_{fi} = \frac{ie^2}{q^2 + i\epsilon} J^{\mu}_{lep}(0) J_{had,\mu}(0) \tag{18}$$

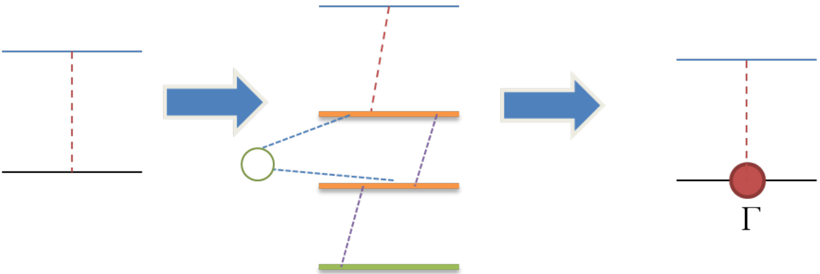
Spin averaged cross section reads

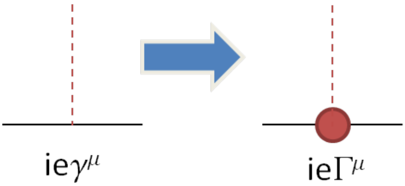
$$\frac{d\sigma}{d\Omega}_{LAB} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot \left[1 - \frac{q^2}{2M^2} \tan^2 \frac{\theta}{2}\right], \tag{19}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2 \cos^2 \frac{\theta}{2} E'}{4E^3 \sin^4 \frac{\theta}{2}} \tag{20}$$

Proton is **Not** Elementary Particle!





$$ie\gamma^{\mu} \rightarrow ie\Gamma^{\mu} \tag{21}$$

$$J^{\mu}_{had}(x) \rightarrow \overline{u}_{proton}(h', r')\Gamma^{\mu} u_{proton}(h, r)e^{-ix\cdot(h'-h)}, \tag{22}$$

Hadronic Current

- Lorentz invariant vector
- Conserved $q_{\mu} J^{\mu} = 0$
- Hermitian, because $H^{\dagger}_{int} = H_{int}$

Gamma-vertex

- $$\Gamma^{\mu} = \gamma^{\mu} F_1(Q^2) + \frac{i\sigma^{\mu\nu} q_{\nu}}{2M} F_2(Q^2). \tag{23}$$
- F_1 and F_2 are real!

Dirac (F_1), and Pauli (F_2) form-factors

$$F_1(0) = 1, \quad F_2(0) = \mu_p - 1 \tag{24}$$

Sachs Form Factors

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \tag{25}$$

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2), \tag{26}$$

where

$$G_E(0) = 1, \quad G_M(0) = \mu_p \tag{27}$$

Finally we have obtained the well known Rosenbluth's formula (Phys. Rev. **79**, 615 (1950)):

$$\frac{d\sigma}{d\Omega}_{LAB} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot \left[G_E^2 + \frac{\tau}{\epsilon} G_M^2\right] \left(\frac{1}{1+\tau}\right), \tag{28}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2 \cos^2 \frac{\theta}{2} E'}{4E^3 \sin^4 \frac{\theta}{2}}, \quad \epsilon = \left(1 + 2(1+\tau) \tan^2 \frac{\theta}{2}\right)^{-1}. \tag{29}$$

Alternative formula:

$$\frac{d\sigma}{d\Omega}_{LAB} = \frac{\alpha^2 E'}{4E^3 \sin^4 \frac{\theta}{2}} \cdot \left[\cos^2 \frac{\theta}{2} \left(F_1^2 - \frac{q^2}{4M^2} F_2^2\right) - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} (F_1 + F_2)^2\right] \tag{30}$$

Reduced Cross Section and Rosenbluth Separation

$$\sigma_R(Q^2, \epsilon) = G_E^2 + \frac{\tau}{\epsilon} G_M^2, \quad \text{or} \quad \sigma_R(Q^2, \epsilon) = \epsilon G_E^2 + \tau G_M^2 \tag{31}$$

ϵ is varied, and Q^2 is fixed, then G_E and G_M are independently extracted.

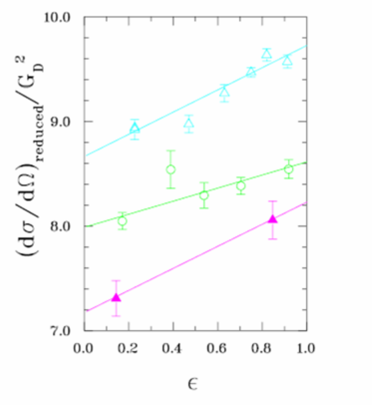


Figure: Taken from C. Perdrisat, V. Punjabi and M. Vanderhaeghen, Prog. Part. Nucl. Phys. **59** (2007) 694.

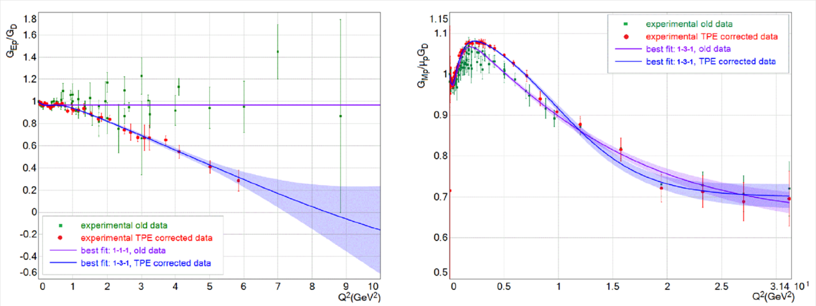
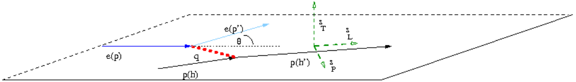


Figure: Taken from K.M. Graczyk, R. Płoński, R. Sulej, JHEP (2010) 053

The form factors can be extracted from the Polarization Transfer data: polarized electron scattering off polarized target, beam asymmetry measurements, etc.



EXAMPLE:

$$\vec{e}(p, s_i) + p(h) \to e(s') + \vec{p}(h', s_f), \tag{32}$$

$$P_L = -G_M^2 I_0 \left(E + E'\right) \sqrt{\tau(1+\tau)} \tan^2 \frac{\theta}{2}, \tag{33}$$

$$P_P = -2 G_E G_M I_0 \sqrt{\tau(1+\tau)} \tan \frac{\theta}{2}, \quad I_0^{-1} = G_E^2 + \frac{\tau}{\epsilon} G_M^2 \tag{34}$$

Hence

$$\frac{G_E}{G_M} = \frac{P_P}{P_L} \frac{E + E'}{2M} \tan \frac{\theta}{2} \tag{35}$$

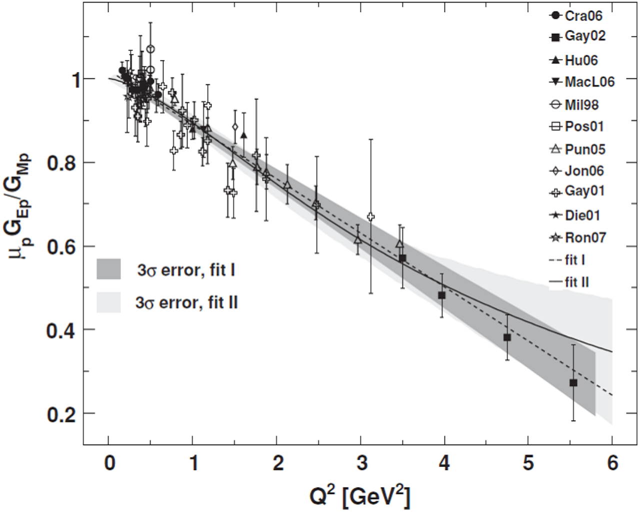


Figure: Taken from W.M. Alberico, C. Giunti, S.M. Bilenky, K.M. Graczyk, Phys. Rev. C**79**, 065204 (2009) 053

Breit Frame

For each Q^2 transfer there exists always the frame with $\nu = 0$,

$$\nu_B = 0, \quad \Rightarrow q^2 = -\mathbf{q}_B^2 \tag{36}$$

In the elastic ep scattering it is the Central Mass frame, where

$$\mathbf{p}_B = \frac{\mathbf{q}_B}{2}, \quad \mathbf{h} = -\frac{\mathbf{q}_B}{2}. \tag{37}$$

The hadronic current reads

$$J^\mu_{had} \quad = \quad \chi^\dagger_{s'} \left(2MG_E(Q^2),\ i\tau \times \mathbf{q}_B G_M(Q^2)\right) \chi_s. \tag{38}$$

In the classical electrodynamics

$$J^{NR} = \left(e\rho^{NR}_q,\ \mu\vec{\sigma} \times \vec{\bigtriangledown}\rho^{NR}_\mu\right) \tag{39}$$

where ρ^{NR}_q and ρ^{NR}_μ is the nonrelativistic (NR) charge and magnetization densities respectively.

$$e = \frac{1}{2M}\int d^3r \left[J^0_{had}\right]_{Breit}(r) \quad (40)$$

$$= \frac{1}{2M(2\pi)^3}\int d^3r \int d^3q e^{-i\mathbf{q}\cdot r}\left[J^0_{had}\right]_{Breit}(\mathbf{q}) \quad (41)$$

$$= \int d^3r \rho_e(r) \quad (42)$$

Hence

$$\rho_e(r) = \frac{1}{(2\pi)^3}\int d^3q e^{-i\mathbf{q}\cdot r} G_E(\mathbf{q}) \quad (43)$$

Indeed

$$\int d^3r \rho_e(r) = \frac{1}{(2\pi)^3}\int d^3r d^3q G_E(\mathbf{q}^2) e^{-i\mathbf{q}\cdot\mathbf{r}} \quad (44)$$

$$= \int d^3q G_E(\mathbf{q}^2)\delta^3(\mathbf{q}) = G_E(0) = 1. \quad (45)$$

$\langle r^2 \rangle$ of Proton: **Standard textbook derivation (in th Breit frame)**

$$G_E(\mathbf{q}^2) = \int d^3 r e^{-i\mathbf{q}\cdot r}\rho(r) = 4\pi \int_0^\infty dr \frac{r}{|\mathbf{q}|} \sin(r|\mathbf{q}|)\rho(r). \quad (46)$$

Assume that $r|\mathbf{q}| \ll 1$, then sin(...) is expanded in the Taylor series,

$$\begin{aligned} G_E(\mathbf{q}^2) &= 4\pi \int_0^\infty dr r^2 \frac{1}{r|\mathbf{q}|} \left(\frac{r|\mathbf{q}|}{1!} - \frac{(r|\mathbf{q}|)^3}{3!} + \ldots \right) \rho(r) \\ &= 4\pi \int_0^\infty dr r^2 \left(1 - \frac{r^2}{6}|\mathbf{q}|^2 + \ldots \right) \rho(r) \quad (47) \\ &= 1 - \frac{|\mathbf{q}|^2}{6} \cdot 4\pi \int_0^\infty dr r^2 \frac{r^2}{6} \rho(r) + \ldots \quad (48) \\ &= 1 + \sum_{i=1}^{\infty} (-1)^i \frac{|\mathbf{q}|^{2i}}{(2i+1)!} \left\langle r^{2i} \right\rangle \quad (49) \end{aligned}$$

In the low-q^2 approximation, the slope of the form factor at $Q^2 = 0$ gives:

$$\left\langle r^2 \right\rangle = -6 \frac{dG_E(\mathbf{q}_B)}{d|\mathbf{q}|^2}\bigg|_{|\mathbf{q}|^2=0} = -6 \frac{dG_E(0)}{dQ^2}\bigg|_{Q^2=0}. \quad (50)$$

- If ρ_e spherically symmetric in the BF, then at low-Q^2 G_E must depend only on even power of $Q = \sqrt{Q^2}$.
- The derivation is valid as long as $|\mathbf{q}|$ is low, and $\rho(\mathbf{r})$ is spherically symmetric. However, in the Breit frame there is one direction distinguished by transfer of momentum $\mathbf{q}$. It means that the $\rho(\mathbf{r})$ **does not to be symmetric**! It seems to be reasonable to assume that only in the rest frame of the nucleon the charge distribution should has spherically symmetric form.
- The problem has been discussed by Kelly (**Phys. Rev. C 66 (2002) 065203**): the charge, and magnetic densities have an interpretation in the rest frame of the nucleon. However, to relate them with the Sachs form factors one needs to apply *relativistic inversion*: boost the quantities from the rest frame of the nucleon to the Breit frame. It was stressed that there is not unique relation between the Sachs form factors measured by electron scattering at finite Q^2 and the static charge and magnetization densities. *The basic problem is that electron scattering measures transition matrix elements between states of composite system that have different momenta and transition densities between such states are different from the static densities in the rest frame. Furthermore, the boost operator for a composite system depends upon the interactions among its constitutes.*

- Yukawa Distribution:

$$\rho(r) = \frac{1}{4\pi r_0^2}\frac{e^{-r/r_0}}{r}, \quad \sqrt{< r^2 >} = \sqrt{6}r_0, \quad G_E(\mathbf{q}^2) = \frac{1}{1+\mathbf{q}^2 r_0^2} \qquad (51)$$

- Exponential Distribution:

$$\rho(r) = \frac{1}{8\pi r_0^3}e^{-r/r_0}, \quad \sqrt{< r^2 >} = \sqrt{12}r_0, \qquad (52)$$

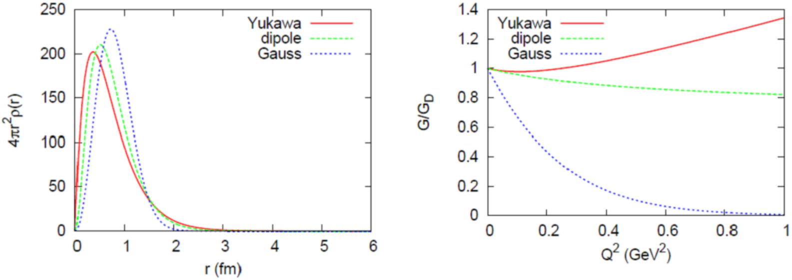
$$G_E(\mathbf{q}^2) = \frac{1}{\left(1+|\mathbf{q}|^2 r_0^2\right)^2} \equiv G_D(\mathbf{q}^2) = \frac{1}{\left(1+\frac{\mathbf{q}^2}{M_V^2}\right)^2} \qquad (53)$$

- Gaussian Distribution (characteristic for the harmonic oscillator models):

$$\rho(r) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^3}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad \sqrt{\langle r^2\rangle} = \sqrt{3}\sigma \qquad (54)$$

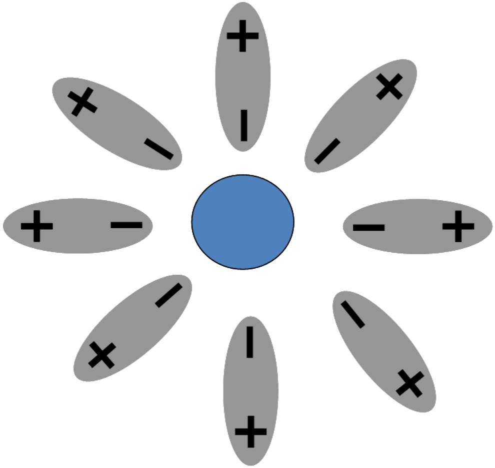
$$G_E(\mathbf{q}^2) = \exp(-\mathbf{q}^2\sigma^2). \qquad (55)$$

Various Charge Distributions



Computed with $\sqrt{\langle r^2 \rangle} = 0.88$ fm.

Proton re-interaction with vacuum



$$H_{NR-QM} = \sum_{i=1}^{3} \left(m_i + \frac{p_i^2}{2m_i} \right) + \frac{1}{2} \sum_{i,j} V(\vec{r}_{ij}), \tag{56}$$

where $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$. Indexes i, j denote the number of the valence quark. In the simplest possible model: $V(\vec{r}_{ij}) = \frac{K}{2} r_{ij}^2$ and $m = m_u = m_d$. It is convenient to work in the center of mass frame CMF. Introduce the the new variables:

$$\vec{R} = \frac{1}{3} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3), \quad \vec{\rho} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2), \quad \vec{\lambda} = \frac{1}{\sqrt{6}} (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3). \tag{57}$$

Then

$$H_{NR-QM} = 3m + \frac{\vec{p}_R^{\,2}}{6m} + \frac{\vec{p}_\rho^{\,2}}{2m} + \frac{\vec{p}_\lambda^{\,2}}{2m} + \frac{3K}{2} \left(\vec{\lambda}^2 + \vec{\rho}^2 \right). \tag{58}$$

At the center-of-mass frame $\vec{p}_R = 0$, then the dynamical part of the hamiltonian reads

$$H_{NR-QM}|_{dynmical} = \underbrace{\frac{\vec{p}_\rho^{\,2}}{2m} + \frac{3K}{2} \vec{\rho}^2}_{\textit{first oscillator}} + \underbrace{\frac{\vec{p}_\lambda^{\,2}}{2m} + \frac{3K}{2} \vec{\lambda}^2}_{\textit{second oscillator}}. \tag{59}$$

The spatial wave function of the proton reads

$$\Phi_0(\vec{\rho},\vec{\lambda}) = \left(\frac{m\omega_0}{\pi}\right)^3 \exp\left(-m\omega_0 \frac{(\vec{\rho}^2+\vec{\lambda}^2)}{2}\right), \quad \omega_0 = \sqrt{\frac{3K}{m}} \tag{60}$$

The proton radius:

$$\langle r^2\rangle_{proton} = \frac{1}{m\omega_0}. \tag{61}$$

Let assume that $m \approx 0.34$ GeV. In the ground level there are two states: nucleon and $\Delta(1232)$. The averaged mass reads $\overline{M} = (M_N + M_\Delta)/2 = 1.1$ GeV. In the next level of baryons lader there are five states with averaged mass $\overline{M}^* = 1.6$ GeV. Hence

$$\omega_0 = 0.5 \text{ GeV}. \tag{62}$$

Hence

$$\langle r^2\rangle_{proton} = 5.9 \text{ GeV}^{-2} = 0.25 \text{ fm}^2 \quad \Rightarrow \quad \sqrt{\langle r^2\rangle_{proton}} = 0.5 \text{ fm}. \tag{63}$$

In reality $\sqrt{\langle r^2\rangle_{proton}} \sim 0.88$ fm.