Elastic ep scattering and higher radiative corrections
Part II

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Motivation

Jeśli nie przestaniecie udowadniać tego, co już zrobili inni, nabierać pewności, komplikować rozwiązań – po prostu dla przyjemności – wtedy, pewnego dnia rozejrzycie się, i stwierdzicie, że tego jeszcze nikt nie zrobił! To jest sposób zostania uczonym!

R. P. Feynman.

There is a systematic discrepancy between ratio $\mu_p G_E/G_M$ data extracted from PT and cross section measurements!

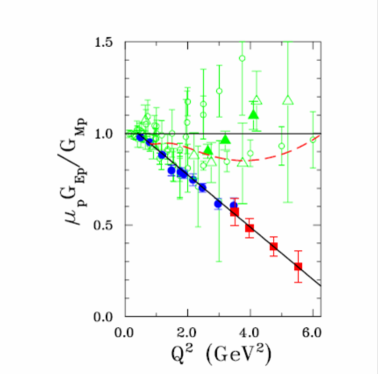
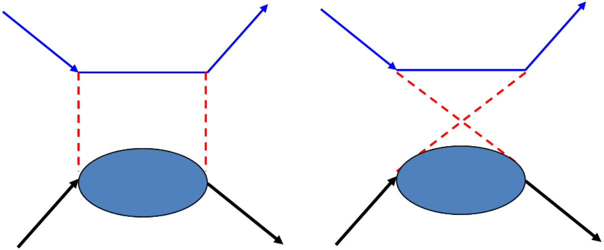


Figure: Taken from C. Perdrisat, V. Punjabi and M. Vanderhaeghen, Prog. Part. Nucl. Phys. **59** (2007) 694.

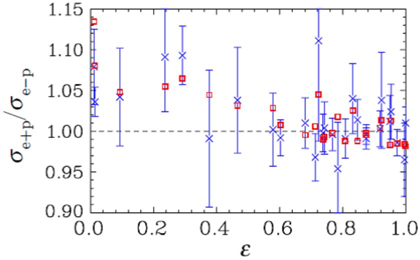
- **The two-photon exchange (TPE) correction (Born-like) is responsible for that!**
- The PT data is less affected by TPE correction than cross section measurements! P. A. M. Guichon and M. Vanderhaeghen, Phys. Rev. Lett. **91** (2003) 142303, P. G. Blunden, W. Melnitchouk and J. A. Tjon, Phys. Rev. Lett. **91** (2003) 142304.



- At least two new experiments dedicated to the investigation of the TPE contribution!

$$\frac{\sigma(e^+ p \to e^+ p)}{\sigma(e^- p \to e^- p)} \approx 1 - \frac{2\Delta C_{2\gamma}}{\sigma_{1\gamma}}. \tag{1}$$

  - J. Arrington *et al.*, *Two-photon exchange and elastic scattering of electrons / positrons on the proton. (Proposal for an experiment at VEPP-3)*, arXiv:nucl-ex/0408020;
  - Jefferson Lab experiment E04-116, *Beyond the Born Approximation: A Precise Comparison of e^+p and e^-p Scattering in CLAS*, W. K. Brooks, *et al.*, spokespersons.



taken from J. Arrington, Phys. Rev. C**71**, 015202 (2005)

- The Proton Radius is extracted from CODATA, as it has been already explained, $\sqrt{\langle r^2 \rangle} = 0.8768 \pm 0.0069$ fm (P. J. Mohr, B. N. Taylor and D. B. Newell, Rev. Mod. Phys. 80, 633 (2008).)
- Lamb shift in muonic atom, $\sqrt{\langle r^2 \rangle} = 0.84184 \pm 0.00067$ fm, R. Pohl, A. Antognini, F. Nez et al., Nature 466, 213 (2010).
- The results are 5σ away of each other!
- The Lamb shift is a small difference in energy between two energy levels ${}^2S^{1/2}$ and ${}^2P_{1/2}$ of the hydrogen atom. According to Dirac, the ${}^2S_{1/2}$ and ${}^2P_{1/2}$ orbitals should have the same energies. However, the interaction between the electron and the vacuum causes a tiny energy shift on ${}^2S_{1/2}$. (see e.g. K. Pachucki, Phys. Rev. A60 (1999) 3593.)

$$L_{exp} = 206.2949 \pm 0.0032 \text{ meV} \qquad (2)$$

$$L_{th} = 209.9779(49) - 5.2262\sqrt{\langle r^2 \rangle} + 0.00913\sqrt{\langle r^3 \rangle_{(2)}} \qquad (3)$$

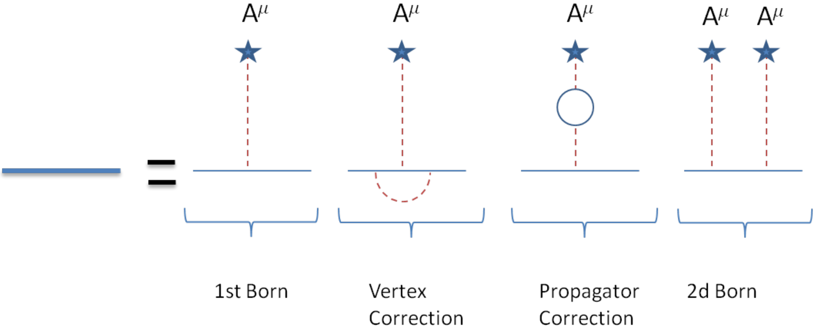
where $\langle r^3 \rangle_{(2)}$ is the third Zemach moment defined as:

$$\langle r^3 \rangle_{(2)} = \int d^3r d^3r' |\mathbf{r} - \mathbf{r}'|^3 \rho(\mathbf{r}')\rho(\mathbf{r}) \qquad (4)$$

$$
\begin{aligned}
\sqrt{\langle r^2 \rangle}_{LAMB} &= 0.84184 \pm 0.00067 \text{ fm} && (5)\\
\sqrt{\langle r^2 \rangle}_{CODATA} &= 0.8768 \pm 0.0069 \text{ fm} && (6)\\
\sqrt{\langle r^2 \rangle}_{dipole} &= 0.81 \text{ fm} && (7)\\
\sqrt{\langle r_E^2 \rangle}_{NN} &= 0.85 \text{ fm} && (8)\\
\sqrt{\langle r_M^2 \rangle}_{NN} &= 0.82 \text{ fm} && (9)\\
& && (10)
\end{aligned}
$$

NN from K.M. Graczyk, Phys. Rev. C 84, 034314 (2011).

Electron Scattering off Coulomb Potential



Our attention is concentrated on the first and second order Born diagrams.

$$|\mathcal{M}|^2 \approx \underbrace{|\mathcal{M}^{(1)}|^2}_{\alpha^2} + \underbrace{2\mathrm{Re}\left(\mathcal{M}^{(1)*}\mathcal{M}^{(2)}\right)}_{\alpha^3} + \underbrace{|\mathcal{M}^{(2)}|^2}_{\alpha^4} \quad (11)$$

$$\frac{d\sigma_{cou.}}{d\Omega} = \frac{|\mathbf{p}'|}{16\pi^2|\mathbf{p}|}\cdot\frac{1}{2}\sum_{spin}|\mathcal{M}|^2 \approx \frac{d\sigma^{(1)}_{cou.}}{d\Omega} + \frac{d\sigma^{(2)}_{cou.}}{d\Omega} + \frac{d\sigma^{(3)}_{cou.}}{d\Omega} \quad (12)$$

$$\langle p'|iT^{(1)}_{fi}|p\rangle = \langle p' \mid -i\int d^4xe\overline{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x) \mid p\rangle \quad (13)$$

$$= -ie\overline{u}(p')\gamma^{\mu}u(p)\int d^4xe^{i(p'-p)x}A_{\mu}(x) \quad (14)$$

$$= -ie\overline{u}(p')\gamma^{\mu}u(p)\tilde{A}_{\mu}(p'-p) \quad (15)$$

$$= -(2\pi i)\delta(E_f - E_i)e\overline{u}(p')\gamma^{\mu}u(p)\tilde{A}_{\mu}(\mathbf{p}' - \mathbf{p}) \quad (16)$$

where we have assumed that A_{μ} is time independent, and $\tilde{A}_{\mu}(p'-p)$ is the Fourier transform:

$$\tilde{A}_{\mu}(p'-p) = (2\pi)\delta(E_f - E_i)\tilde{A}_{\mu}(\mathbf{p}' - \mathbf{p}), \quad (17)$$

and

$$\tilde{A}_{\mu}(\mathbf{p}' - \mathbf{p}) = \int d^3re^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}A_{\mu}(\mathbf{r}). \quad (18)$$

T-Matrix: 2st Born

$$\langle p' \mid iT^{(2)} \mid p\rangle = -\frac{1}{2}\langle p' \mid \int d^4x d^4y e^2 \overline{\psi}(x)\gamma^\mu \Psi(x) A_\mu(x) \overline{\psi}(y)\gamma^\nu \Psi(y) A_\nu(y) \mid p\rangle \tag{19}$$

$$= -ie^2 \int \frac{d^4l}{(2\pi)^4} \frac{\overline{u}(p')\gamma^\mu(\hat{l}+m_e)\gamma^\nu u(p)}{l^2 - m_e^2 + i\epsilon} \tilde{A}_\mu(p'-l)\tilde{A}_\nu(l-p) \tag{20}$$

Coulomb Potential and M-matrix

Point-like Coulomb potential

$$A^{\mu}_{poin}(\vec{r}) = g^{\mu 0}\frac{Ze}{4\pi|\mathbf{r}|}. \quad (21)$$

$$\begin{aligned}\tilde{A}^{0}_{poin}(\mathbf{p}'-\mathbf{p}) &= \int d^3 r e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\frac{Ze}{4\pi|\mathbf{r}|} = \frac{Ze}{4\pi\mathbf{q}^2}\int d^3 r \triangle\left(e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\right)\frac{1}{|\mathbf{r}|} \quad (22)\\ &= \frac{Ze}{4\pi\mathbf{q}^2}\int d^3 r e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\triangle\left(\frac{1}{|\mathbf{r}|}\right) = -\frac{Ze}{\mathbf{q}^2}. \quad (23)\end{aligned}$$

Hence

$$\langle p'|iT^{(1)}|p\rangle = (2\pi i)\delta(E_f - E_i)\frac{Ze^2}{\mathbf{q}^2}\overline{u}(p')\gamma^0 u(p). \quad (24)$$

$$\langle p'|iT^{(2)}|p\rangle = -(2\pi i)\delta(p'_0 - p_0)Z^2 e^4\int\frac{d^3 l}{(2\pi)^3}\frac{\overline{u}(p')(\gamma^0 E + \vec{\gamma}\cdot\mathbf{l} + m_e)u(p)}{(\mathbf{l}^2 - \mathbf{p}^2 + i\epsilon)(\mathbf{p}' - \mathbf{l})^2(\mathbf{l} - \mathbf{p})^2}, \quad (25)$$

where we have integrated the energy component of dl_0, namely

$$\int\frac{dl_0}{2\pi}(2\pi)\delta(p'_0 - l_0)(2\pi)\delta(l_0 - p_0) = 2\pi\delta(p'_0 - p_0).$$

and we substituted, $l_0 = E$ as well as $l^2 = l_0^2 - \mathbf{l}^2 - m_e^2 = \mathbf{p}^2 - \mathbf{l}^2$. Hence

$$\gamma^0(\hat{l} + m_e)\gamma^0 = \gamma^0(\gamma^0 l^0 - \vec{\gamma}\cdot\mathbf{l} + m_e)\gamma^0 = \gamma^0 l^0 + \vec{\gamma}\cdot\mathbf{l} + m_e \to \gamma^0 E + \vec{\gamma}\cdot\mathbf{l} + m_e \quad (26)$$

In order to compute the cross section one has to compute the $\mathcal{M}$ matrix,

$$\langle p' \mid iT \mid p \rangle = \mathcal{M}(2\pi i)\delta({p'}_0 - p_0). \quad (27)$$

Similarly as in the case of the T matrix, the $\mathcal{M}$ matrix can be written in as the perturbative series:

$$\mathcal{M} = \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots \quad (28)$$

- First order:

$$\mathcal{M}^{(1)} = \frac{Ze^2}{\mathbf{q}^2}\overline{u}(p')\gamma^0 u(p). \quad (29)$$

- Second order:

$$\mathcal{M}^{(2)} = -Z^2 e^4 \int \frac{d^3 l}{(2\pi)^3} \frac{\overline{u}(p')(\gamma^0 E + \vec{\gamma}\cdot\mathbf{l} + m_e)u(p)}{(\mathbf{p}^2 - \mathbf{l}^2 + i\epsilon)(\mathbf{p}' - \mathbf{l})^2(\mathbf{l} - \mathbf{p})^2}. \quad (30)$$

Suppose that the Coulomb potential is not point-like but has its own spherical distribution.

$$A^\mu(\vec{r}) = g^{\mu 0}\phi(\mathbf{r}), \quad \phi(\mathbf{r}) = \frac{Z}{4\pi}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \quad Z = \int d^3r\rho(\mathbf{r}) \tag{31}$$

The Fourier transformation of the potential reads

$$\begin{aligned} A^0(\mathbf{q}) &= \frac{Ze}{4\pi}\int d^3r d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{Ze}{4\pi\mathbf{q}^2}\int d^3r d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}}\triangle_r \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{Ze}{\mathbf{q}^2}\int d^3r d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r}')\delta^{(3)}(\mathbf{r}-\mathbf{r}') \\ &= \frac{Ze}{\mathbf{q}^2}\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r}) \equiv \frac{Ze}{\mathbf{q}^2}\underbrace{F(\mathbf{q})}_{\textit{form factor}} . \end{aligned} \tag{32}$$

$$F(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r}) \quad \rho(\mathbf{r}) = \int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}F(\mathbf{q}). \tag{33}$$

$$\frac{d\sigma_{1\gamma}}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2\mathbf{p}^2\sin^4\left(\frac{\theta}{2}\right)}\left(1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right), \tag{34}$$

where

$$\beta^2 = \frac{|\mathbf{p}^2|}{E^2} = v^2_{electron}. \tag{35}$$

Notice that for $\beta \to 0$ we have well known Mott scattering formula,

$$\frac{d\sigma^{(1)}_{coul.}}{d\Omega} \approx \frac{Z^2\alpha^2}{4\beta^2\mathbf{p}^2\sin^4\left(\frac{\theta}{2}\right)}. \tag{36}$$

Notice that if instead of the point-like potential the one given by (31) is discussed the cross section in the first order Born approximation reads

$$\frac{d\sigma^{(1)}}{d\Omega} = \frac{\alpha^2}{4\beta^2\mathbf{p}^2\sin^4\left(\frac{\theta}{2}\right)}\left(1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right)F^2(\mathbf{q}). \tag{37}$$

- Point-like static potential:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E}{4E^3 \sin^4\left(\frac{\theta}{2}\right)}. \tag{38}$$

- Spatial-charge distribution but still static:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E}{4E^3 \sin^4\left(\frac{\theta}{2}\right)} F^2(\mathbf{q}^2). \tag{39}$$

- Proton with spin 1/2:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E'}{4E^3 \sin^4 \frac{\theta}{2}} \cdot \left[\cos^2 \frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2 \frac{\theta}{2}\right], \tag{40}$$

(recoil correction, spin 1/2 correction)

- Proton with spin 1/2, and magnetic anomalous moment

$$\frac{d\sigma}{d\Omega}_{LAB} = \frac{\alpha^2 E'}{4E^3 \sin^4 \frac{\theta}{2}} \cdot \left[\cos^2 \frac{\theta}{2} \left({F_1}^2 + \frac{Q^2}{4M^2} {F_2}^2\right) + \frac{Q^2}{2M^2} \sin^2 \frac{\theta}{2} (F_1 + F_2)^2\right] \tag{41}$$

$(\gamma^\mu, \sigma^{\mu\nu})$.

Notice that for $Q^2 \to 0$, $\sin^2\frac{\theta}{2} \to 0$, $\cos^2\frac{\theta}{2} \to 1$ the static potential $F(\mathbf{q}^2)$ form factor has the same meaning as $F_1(Q^2)$, and G_E because $G_E = F_1 - \tau F_2$.

$$\mathcal{M}^{(2)} = -Z^2e^4 \int \frac{d^3l}{(2\pi)^3} \frac{\overline{u}(p')(\gamma^0 E + \vec{\gamma}\cdot\mathbf{l} + m_e)u(p)}{(\mathbf{p}^2 - \mathbf{l}^2 + i\epsilon)(\mathbf{p}' - \mathbf{l})^2(\mathbf{l} - \mathbf{p})^2} \quad (42)$$

$$= Z^2e^4\overline{u}(p')\left(\gamma_0 E \mathrm{I}_1 + \vec{\gamma}\cdot\mathbf{I}_2\right)u(p) \quad (43)$$

- Long-range character of the Coulomb forces $\rightarrow$ infrared singularities
- One has to extract the divergent term from the amplitude
- R. H. Dalitz Proc. R. Soc. Lond. A**206**, 509 (1951).

$$\frac{1}{\mathbf{q}^2} \rightarrow \frac{1}{\mathbf{q}^2 + \mu^2}. \quad (44)$$

It corresponds to the screened Coulomb interaction

$$A^\mu(r) = g^{\mu 0}\frac{Zeexp(-\mu|\mathbf{r}|)}{4\pi|\mathbf{r}|} = -\int \frac{d^4q}{(2\pi)^3}\frac{\delta(q_0)}{q^2 - \mu^2}e^{iq\cdot r} \quad (45)$$

The problem is seen already in QM, if the potential $V(r)$ does not converge faster then $1/r$ then the partial wave solution can not be obtained. For potential $1/r$ distorted wave functions are obtained!

Properties of the integrals I_1 and $\mathbf{I}_2$

Notice that I_1 and $\mathbf{I}_2$ are symmetric under exchange $\mathbf{p} \leftrightarrow \mathbf{p}$,

$$\mathrm{I}_1 = \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)(\mathbf{p}' - \mathbf{l})^2 (\mathbf{l} - \mathbf{p})^2}, \quad \mathbf{I}_2 = \int \frac{d^3 l}{(2\pi)^3} \frac{\mathbf{l}}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)(\mathbf{p}' - \mathbf{l})^2 (\mathbf{l} - \mathbf{p})^2} \tag{46}$$

hence $\mathbf{I}_2 \sim \mathbf{p} + \mathbf{p}'$. Notice that $\overline{u}(p', s')\vec{\gamma} u(p, s) = \chi^\dagger_{s'}(\mathbf{p} + \mathbf{p}' + i\mathbf{q} \times \tau)\chi_s$.

Feynman's Identity

$$\frac{1}{(a+\lambda)(b+\lambda)} = -\frac{\partial}{\partial \lambda} \int_0^1 d\alpha \frac{1}{\alpha a + (1-\alpha) b + \lambda} = -\int_0^1 d\alpha \frac{\partial}{\partial(\alpha a)} \frac{1}{\alpha a + (1-\alpha) b + \lambda} \tag{47}$$

Then

$$\mathrm{I}_1 = -\int_0^1 d\alpha \frac{\partial}{\partial \mu^2} \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)((\mathbf{l} - \mathbf{P})^2 + M_0^2)} \tag{48}$$

$$= -\int_0^1 d\alpha \frac{\partial}{\partial M_0^2} \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)((\mathbf{l} - \mathbf{P})^2 + M_0^2)} \tag{49}$$

$$\mathrm{I}_2^k = \int_0^1 d\alpha \left(\frac{\partial}{2\partial P^k} - P^k \frac{\partial}{\partial M_0^2} \right) \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)((\mathbf{l} - \mathbf{P})^2 + M_0^2)} \tag{50}$$

where

$$\mathbf{P} = \alpha \mathbf{p} + (1-\alpha)\mathbf{p}', \quad M_0^2 = \mu^2 + 4\alpha(1-\alpha)\mathbf{q}^2 \tag{51}$$

Properties of the integrals I_1 and $\mathbf{I}_2$

$$\mathrm{I}_1 = -\int_0^1 d\alpha \frac{\partial}{\partial M_0^2} I \tag{52}$$

$$\mathrm{I}_2^k = \int_0^1 d\alpha \left(\frac{\partial}{2\partial P^k} - P^k \frac{\partial}{\partial M_0^2} \right) I \tag{53}$$

$$\mathrm{I} = \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)((\mathbf{l} - \mathbf{P})^2 + M_0^2)} \tag{54}$$

Notice that

$$\mathrm{I} \sim \int \frac{d^3 l}{l^4}$$

$|\mathcal{M}|^2$

$$|\mathcal{M}|^2 \approx |\mathcal{M}^{(1)}|^2 + \underbrace{2\,\mathrm{Re}\left(\mathcal{M}^{(1)*}\mathcal{M}^{(2)}\right)}_{*} + |\mathcal{M}^{(2)}|^2, \tag{55}$$

$$\begin{aligned}
\sum_{spin} * &= \frac{Z^3 e^6}{|\mathbf{q}|^2} \sum_{spin} \left(\overline{u}(p')\gamma^0 u(p)\right)^* \left\{ \mathrm{I}_1 E \overline{u}(p')\gamma^0 u(p) + \mathbf{I}_2 \cdot \overline{u}(p')\vec{\gamma} u(p) \right\} \\
&= \frac{Z^3 e^6}{|\mathbf{q}|^2} \left[\mathrm{I}_1 E \,\mathrm{Tr}(\hat{p}\gamma^0 \hat{p}'\gamma^0) + \sum_{k=1}^{3} \mathrm{I}_2^k \,\mathrm{Tr}(\hat{p}\gamma^0 \hat{p}'\gamma^k) \right] \\
&= \frac{Z^3 e^6}{|\mathbf{q}|^2} \left[4\mathrm{I}_1 E^3 (1 + \beta^2 \cos\theta) + 4E\mathbf{I}_2 \cdot (\mathbf{p} + \mathbf{p}') \right]
\end{aligned} \tag{56}$$

$$= \frac{4Z^3 e^6 E^3 \cos^2\frac{\theta}{2}}{\mathbf{q}^2} \left[\frac{(1 + \beta^2 \cos\theta)}{\cos^2\frac{\theta}{2}} \mathrm{I}_1 + \frac{\mathbf{I}_2 \cdot (\mathbf{p} + \mathbf{p}')}{E^2 \cos^2\frac{\theta}{2}} \right] \tag{57}$$

$$\mathrm{Tr}\hat{p}\gamma^0\hat{p}'\gamma^0 = 4\left[2E_iE_f - p\cdot p'\right] \quad (58)$$

$$= 4(E^2 + \mathbf{p}\cdot\mathbf{p}') = 4E^2(1+\beta^2\cos\theta) \quad (59)$$

$$\mathrm{Tr}\hat{p}\gamma^0\hat{p}'\gamma^k = 4\left[p^kE + p'^kE\right] \quad (60)$$

$$\sum_{k=1}^{3}\mathrm{I}^k\mathrm{Tr}\hat{p}\gamma^0\hat{p}'\gamma^k = 4E\mathbf{I}_2\cdot(\mathbf{p}+\mathbf{p}') \quad (61)$$

$$(62)$$

$$(\mathbf{p}+\mathbf{p}')^2E = 2|\mathbf{p}|(\mathbf{p}^2+\mathbf{p}\cdot\mathbf{p}') = 4\frac{|\mathbf{p}|^3}{\beta}\cos^2\frac{\theta}{2} \quad (63)$$

$$\mathbf{p}^2 + 2\mathbf{p}\cdot\mathbf{p}' = |\mathbf{p}|^2(4\cos^2\frac{\theta}{2} - 1) \quad (64)$$

I

$$
\begin{aligned}
\mathrm{I}(p^2, \mathbf{P}, M_0^2) &= \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\mathbf{l}^2 - \mathbf{p}^2 - i\epsilon)((\mathbf{l} - \mathbf{P})^2 + M_0^2(\mu^2))} && (65) \\
&= \int_{-1}^{1} dt \int_0^\infty \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2 - p^2 - i\epsilon)} \frac{1}{l^2 - 2tPl + P^2 + M_0^2} && (66) \\
&= \frac{1}{2} \int_{-1}^{1} dt \int_0^\infty \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2 - p^2 - i\epsilon)} \frac{1}{l^2 - 2tPl + P^2 + M_0^2} \\
&\quad + \frac{1}{2} \int_{-1}^{1} dt \int_0^\infty \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2 - p^2 - i\epsilon)} \frac{1}{l^2 + 2tPl + P^2 + M_0^2} && (67) \\
&\quad t \to -t, \quad \int_{-1}^{1} dt \to \int_{-1}^{1} dt && (68) \\
& && (69)
\end{aligned}
$$

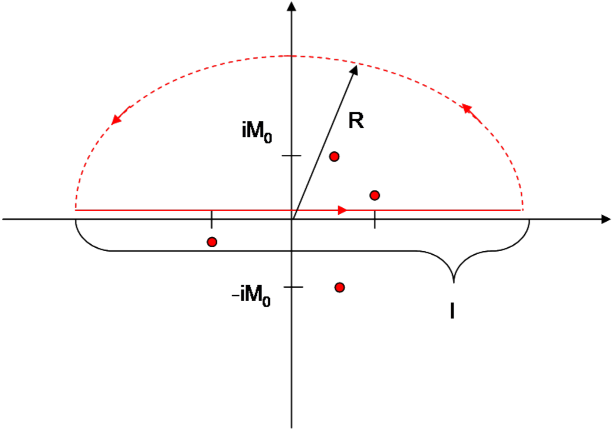
I

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{2}\int_{-1}^{1} dt \int_{0}^{\infty} \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2-p^2-i\epsilon)} \frac{1}{l^2-2tPl+P^2+M_0^2} \\
&+ \frac{1}{2}\int_{-1}^{1} dt \int_{-\infty}^{0} \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2-p^2-i\epsilon)} \frac{1}{l^2-2tPl+P^2+M_0^2}, \quad (70) \\
& l \to -l, \quad \int_{0}^{\infty} dl \to \int_{-\infty}^{0} dl \quad (71) \\
I &= \frac{1}{2}\int_{-1}^{1} dt \int_{-\infty}^{\infty} \frac{dl}{(2\pi)^2} \frac{l^2}{(l^2-p^2-i\epsilon)} \frac{1}{l^2-2tPl+P^2+M_0^2} \quad (72) \\
l^2-p^2-i\epsilon &= (l-p-i\epsilon)(l+p+i\epsilon) \quad (73) \\
&= \frac{1}{2}\int_{-1}^{1} dt \int_{-\infty}^{\infty} \frac{dl}{(2\pi)^2} \frac{l^2}{(l-p-i\epsilon)(l+p+i\epsilon)(l-l_+)(l-l_-)}, \quad (74)
\end{aligned}
$$

where

$$
\begin{aligned}
l_\pm &= Pt \pm i\sqrt{(1-t^2)P^2+M_0^2} \quad (75) \\
M_0^2 &= \mathbf{p}^2+\mu^2-\mathbf{P}^2 = \mu^2+4\alpha(1-\alpha)\mathbf{q}^2 \quad (76)
\end{aligned}
$$

I



$$\underbrace{\int_{-\infty}^{\infty}(\ldots)}_{=\mathrm{I}} + \underbrace{\int_{R}(\ldots)}_{\to 0 \text{ for } R\to\infty} = 2\pi i \sum \mathrm{Res}(\textit{top semicircle}) \tag{77}$$

I

$$\mathrm{I} = \frac{ip}{8\pi}\int_{-1}^{1} dt \frac{1}{p^2 - 2tPp + P^2 + M_0^2} + \frac{1}{8\pi}\int_{-1}^{1} dt \frac{l_+^2}{(l_+^2 - p^2)\left(\sqrt{(1-t^2)P^2 + M_0^2}\right)} \tag{78}$$

$$= \frac{i}{16\pi P}\left[\ln\left(p^2 + 2Pp + P^2 + M_0^2\right) - \ln\left(p^2 - 2Pp + P^2 + M_0^2\right)\right] + \frac{1}{8\pi}\int_{-1}^{1} dt \frac{l_+^2}{(l_+^2 - p^2)\left(\sqrt{(1-t^2)P^2 + M_0^2}\right)}, \tag{79}$$

In the second integral we do the change of the variables, $t \to l_+$, indeed

$$dl_+ = -\frac{iPl_+ dt}{\sqrt{(1-t^2)P^2 + M_0^2}} \tag{80}$$

where

$$\frac{1}{8\pi}\int_{-1}^{1} dt \frac{l_+^2}{(l_+^2 - p^2)\left(\sqrt{(1-t^2)P^2 + M_0^2}\right)} = \frac{i}{8\pi P}\int dl_+ \frac{l_+}{(l_+^2 - p^2)} = \frac{i}{16\pi P}\ln(l_+^2 - p^2) \tag{81}$$

I

$$
\begin{aligned}
\mathrm{I} &= \frac{i}{16\pi P}\left[\ln\left(p^2+2Pp+P^2+M_0^2\right)-\ln\left(p^2-2Pp+P^2+M_0^2\right)\right] \\
&\quad +\frac{i}{16\pi P}\left[\ln((P+iM_0)^2-p^2)-\ln((P-iM_0)^2-p^2)\right], \qquad (82) \\
&= \frac{i}{8\pi P}\ln\left(\frac{p+P+iM_0}{p-P+iM_0}\right) \qquad (83)
\end{aligned}
$$

I_1

$$\mathrm{I}_1 = -\int_0^1 d\alpha \frac{\partial}{2M_0 \partial M_0} \mathrm{I} \tag{84}$$

Hence

$$\begin{aligned}
\mathrm{I}_1 &= \frac{1}{8\pi} \int_0^1 \frac{d\alpha}{2PM_0} \left[\frac{1}{p + P + iM_0} - \frac{1}{p - P + iM_0} \right] \\
&= -\frac{1}{8\pi} \int_0^1 d\alpha \frac{1}{M_0} \frac{1}{(p + iM_0)^2 - P^2} && (85) \\
&= -\frac{1}{8\pi} \int_0^1 d\alpha \frac{1}{M_0} \frac{1}{-\mu^2 + i2pM_0} && (86) \\
&= \frac{1}{8\pi\mu^2} \int_0^1 \frac{d\alpha}{M_0} - \frac{1}{8\pi\mu^2} \int_0^1 d\alpha \frac{2pi}{-\mu^2 + i2pM_0} && (87)
\end{aligned}$$

$$M_0^2 = \mathbf{q}^2\alpha(1-\alpha)+\mu^2 = \left(\frac{\mathbf{q}^2}{4}+\mu^2\right)\left[1-\frac{\mathbf{q}^2}{\frac{\mathbf{q}^2}{4}+\mu^2}\left(\alpha-\frac{1}{2}\right)^2\right] \quad (88)$$

$$\alpha' = \frac{2|\mathbf{q}|}{\sqrt{\mathbf{q}^2+4\mu^2}}\left(\alpha-\frac{1}{2}\right) \quad (89)$$

Now,

$$\int\frac{d\alpha}{M_0} = \frac{1}{|\mathbf{q}|}\int\frac{d\alpha'}{\sqrt{1-\alpha'^2}} = \frac{1}{|\mathbf{q}|}\arcsin(\alpha') = \frac{1}{|\mathbf{q}|}\arcsin\left(\frac{2|\mathbf{q}|}{\sqrt{\mathbf{q}^2+4\mu^2}}\left(\alpha-\frac{1}{2}\right)\right) \quad (90)$$

$$\int_0^1\frac{d\alpha}{M_0} = \frac{2}{|\mathbf{q}|}\arcsin\left(\frac{|\mathbf{q}|}{\sqrt{\mathbf{q}^2+4\mu^2}}\right) = \frac{2}{|\mathbf{q}|}\arcsin\left(\frac{1}{\sqrt{1+\frac{4\mu^2}{\mathbf{q}^2}}}\right) \to \frac{\pi}{|\mathbf{q}|} \quad (91)$$

where

$$\frac{1}{\sqrt{1-x}} \approx 1+\frac{x}{2}, \quad \arcsin(1+x) \approx \frac{\pi}{2}(1+x)$$

I_1

$$\int_0^1 d\alpha \frac{1}{-\mu^2 + i2pM_0} = \frac{1}{2p}\int_0^1 d\alpha \frac{1}{-\frac{\mu^2}{2p} + iM_0} \tag{92}$$

$$= \frac{1}{2p}\int_0^1 d\alpha \frac{1}{-\frac{\mu^2}{2p} + i\sqrt{\mu^2 + \alpha(1-\alpha)\mathbf{q}^2}} \tag{93}$$

$$= \frac{1}{p|\mathbf{q}|}\int_0^{\frac{1}{2}} d\alpha' \frac{1}{-\frac{\mu^2}{2p|\mathbf{q}|} + i\sqrt{\frac{1}{4} + \frac{\mu^2}{\mathbf{q}^2} - \alpha'^2}} \tag{94}$$

$$\alpha' = \alpha - \frac{1}{2} \tag{95}$$

From Mathematica

$$\int \frac{dx}{-B + i\sqrt{A^2 - x^2}} = -i\arctan\left(\frac{x}{\sqrt{A^2 - x^2}}\right) + \tag{96}$$

$$\frac{-iB\tanh^{-1}\left(\frac{Bx}{\sqrt{-A^2-B^2}\sqrt{A^2-x^2}}\right) + B\arctan\left(\frac{x}{\sqrt{-A^2-B^2}}\right)}{\sqrt{-A^2-B^2}}$$

$$* = \int_0^{\frac{1}{2}} d\alpha' \frac{1}{-\frac{\mu^2}{2p|\mathbf{q}|} + i\sqrt{\frac{1}{4} + \frac{\mu^2}{\mathbf{q}^2} - \alpha'^2}} \tag{97}$$

$$= \frac{1}{\mu^2}\left\{ -i \arctan\left(\frac{|\mathbf{q}|}{2\mu}\right) - \frac{i\frac{\mu^2}{2p|\mathbf{q}|}}{\sqrt{\frac{1}{4} + \frac{\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{4p^2\mathbf{q}^2}}} \left[\arctan\left(-\frac{i}{2\sqrt{\frac{1}{4} + \frac{\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{4p^2\mathbf{q}^2}}} \right) - \right.\right.$$

$$\left.\left. -i \tanh^{-1}\left(-\frac{i\mu}{4p}\left[1 + \frac{4\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{p^2\mathbf{q}^2}\right]^{-\frac{1}{2}} \right) \right] \right\} \tag{98}$$

$$= -\frac{i\pi}{2\mu^2} - \frac{1}{p|\mathbf{q}|} \tanh^{-1}\left(\left[1 + \frac{4\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{p^2\mathbf{q}^2}\right]^{-\frac{1}{2}} \right) = -\frac{i\pi}{2\mu^2} - \frac{1}{p|\mathbf{q}|} \ln \frac{p \sin\frac{\theta}{2}}{\mu} \tag{99}$$

$$\tan(-iz) = \frac{1}{i}\tanh(z), \quad \tanh(iz) = i\tan(z)$$

$$\left(1 + \frac{4\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{p^2\mathbf{q}^2}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\left(\frac{4\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{p^2\mathbf{q}^2}\right) \approx 1 - \frac{2\mu^2}{\mathbf{q}^2} \tag{100}$$

$$\tanh^{-1}\left(\left[1 + \frac{4\mu^2}{\mathbf{q}^2} + \frac{\mu^4}{p^2\mathbf{q}^2}\right]^{-\frac{1}{2}} \right) \approx \frac{1}{2}\left[\ln(2) - \ln\left(\frac{2\mu^2}{\mathbf{q}^2}\right)\right] = \frac{1}{2}\ln\frac{\mathbf{q^2}}{\mu^2} = \ln\frac{2p\sin\frac{\theta}{2}}{\mu}$$

$$\mathrm{I}_1 = \frac{1}{8\pi}\left[\frac{\pi}{|\mathbf{q}|\mu^2} - \frac{2i}{|\mathbf{q}|}\left(-\frac{i\pi}{2\mu^2} - \frac{1}{p|\mathbf{q}|}\ln\frac{2p\sin\frac{\theta}{2}}{\mu}\right)\right] \tag{101}$$

$$= \frac{i}{4\pi q^2 p}\ln\frac{2p\sin\frac{\theta}{2}}{\mu} \tag{102}$$

$$= \frac{i}{16\pi\sin^2\frac{\theta}{2}p^3}\ln\frac{2p\sin\frac{\theta}{2}}{\mu} \tag{103}$$

It is divergent when $\mu \to 0$, but does not contribute to the spin averaged interference term $2\mathrm{Re}(\mathcal{M}^{(1)}\mathcal{M}^{(2)*})$

I_2

$$\mathbf{p}\cdot\mathbf{I}_2 = \int \frac{d^3l}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{l}}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{p}'-\mathbf{l})^2+\mu^2)((\mathbf{l}-\mathbf{p})^2+\mu^2)} \tag{104}$$

$$\mathbf{p}'\cdot\mathbf{I}_2 = \int \frac{d^3l}{(2\pi)^3}\frac{\mathbf{p}'\cdot\mathbf{l}}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{p}'-\mathbf{l})^2+\mu^2)((\mathbf{l}-\mathbf{p})^2+\mu^2)}. \tag{105}$$

Notice that

$$\frac{1}{(\mathbf{l}-\mathbf{p})^2+\mu^2}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)} = \frac{2\mathbf{l}\cdot\mathbf{p}-2p^2-\mu^2}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p})^2+\mu^2)} \tag{106}$$

$$\frac{1}{(\mathbf{l}-\mathbf{p}')^2+\mu^2}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)} = \frac{2\mathbf{l}\cdot\mathbf{p}'-2p^2-\mu^2}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p}')^2+\mu^2)} \tag{107}$$

hence

$$\frac{\mathbf{l}\cdot\mathbf{p}}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p})^2+\mu^2)} = \frac{1}{2}\left[\frac{1}{(\mathbf{l}-\mathbf{p})^2+\mu^2}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)}+\frac{2p^2+\mu^2}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p})^2+\mu^2)}\right]$$

$$\frac{\mathbf{l}\cdot\mathbf{p}'}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p}')^2+\mu^2)} = \frac{1}{2}\left[\frac{1}{(\mathbf{l}-\mathbf{p}')^2+\mu^2}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)}+\frac{2p^2+\mu^2}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{l}-\mathbf{p}')^2+\mu^2)}\right]$$

I_2

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{I}_2 &= \frac{1}{2}\int \frac{d^3l}{(2\pi)^3}\frac{1}{((\mathbf{p}'-\mathbf{l})^2+\mu^2)}\left(\frac{1}{(\mathbf{l}-\mathbf{p})^2+\mu^2)}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)}\right)+\left(p^2+\frac{\mu^2}{2}\right)\mathrm{I}_1 \\
&= \frac{1}{2}(A-B_a)+\left(p^2+\frac{\mu^2}{2}\right)\mathrm{I}_1 \qquad (108)\\
\mathbf{p}'\cdot\mathbf{I}_2 &= \frac{1}{2}\int \frac{d^3l}{(2\pi)^3}\frac{1}{((\mathbf{p}-\mathbf{l})^2+\mu^2)}\left(\frac{1}{(\mathbf{l}-\mathbf{p}')^2+\mu^2}-\frac{1}{(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)}\right)+\left(p^2+\frac{\mu^2}{2}\right)\mathrm{I}_1 \\
&= \frac{1}{2}(A-B_b)+\left(p^2+\frac{\mu^2}{2}\right)\mathrm{I}_1 \qquad (109)
\end{aligned}$$

where

$$\begin{aligned}
A &= \int \frac{d^3l}{(2\pi)^3}\frac{1}{((\mathbf{p}'-\mathbf{l})^2+\mu^2)(\mathbf{l}-\mathbf{p})^2+\mu^2)} \qquad (110)\\
B_a &= \int \frac{d^3l}{(2\pi)^3}\frac{1}{((\mathbf{p}'-\mathbf{l})^2+\mu^2)(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)} \qquad (111)\\
B_b &= \int \frac{d^3l}{(2\pi)^3}\frac{1}{((\mathbf{p}-\mathbf{l})^2+\mu^2)(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)} \qquad (112)
\end{aligned}$$

It is easy to show that $B_a = B_b \equiv B$.

A

We use the Feynman's trick,

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2}$$

$$\begin{aligned}
A &= \int \frac{d^3 l}{(2\pi)^3} \frac{1}{((\mathbf{p}' - \mathbf{l})^2 + \mu^2)(\mathbf{l} - \mathbf{p})^2 + \mu^2)} = \int_0^1 dx \int \frac{d^3 l}{(2\pi)^3} \frac{1}{\left[\mathbf{l}^2 - 2\mathbf{l}\cdot(x\mathbf{p} + (1-x)\mathbf{p}') + \mu^2\right]^2} \\
&= \int_0^1 dx \int \frac{d^3 l}{(2\pi)^3} \frac{1}{\left[\mathbf{l}^2 - 2\mathbf{l}\cdot(x\mathbf{p} + (1-x)\mathbf{p}') + p^2 + \mu^2\right]^2}, \quad \mathbf{l}' = \mathbf{l} - (x\mathbf{p} + (1-x)\mathbf{p}') && (113) \\
&= \int_0^1 dx \int \frac{d^3 l'}{(2\pi)^3} \frac{1}{\left[\mathbf{l}'^2 + M_0^2\right]^2}, \quad M_0^2 = \mu^2 + \mathbf{q}^2 x(1-x) && (114) \\
&= \int_0^1 dx \int_{-\infty}^{\infty} \frac{dl}{4\pi^2} \frac{l^2}{\left[l^2 + M_0^2\right]^2} = \frac{1}{8\pi} \int_0^1 \frac{dx}{M_0} = \frac{1}{8\pi} \int_0^1 \frac{dx}{\sqrt{\mu^2 + \mathbf{q}^2 x(1-x)}} && (115) \\
&= \frac{1}{8|\mathbf{q}|} && (116)
\end{aligned}$$

$\mathbf{I}_2$

Analogically w obtain

$$B = \frac{i}{8\pi p}\left(\ln\left(\frac{2p}{\mu}\right) - i\frac{\pi}{2}\right) = \mathrm{I}(p^2, \mathbf{p}', \mu^2) \tag{117}$$

hence

$$(\mathbf{p}+\mathbf{p}')\cdot\mathbf{I}_2 = A - B + 2\mathbf{p}^2 I_1 = \frac{1}{8|\mathbf{q}|} - \frac{i}{8\pi p}\left(\ln\left(\frac{2p}{\mu}\right) - i\frac{\pi}{2}\right) + \frac{ip}{4\pi\mathbf{q}^2}\ln\frac{2p\sin\frac{\theta}{2}}{\mu} \tag{118}$$

$$= \frac{1-\sin\frac{\theta}{2}}{8|\mathbf{q}|} - \underbrace{\frac{i}{16\pi p}\left(2\ln\left(\frac{2p}{\mu}\right) - \frac{1}{\sin^2\frac{\theta}{2}}\ln\frac{2p\sin\frac{\theta}{2}}{\mu}\right)}_{\textit{divergent}} \tag{119}$$

$\sigma^{(2)}$

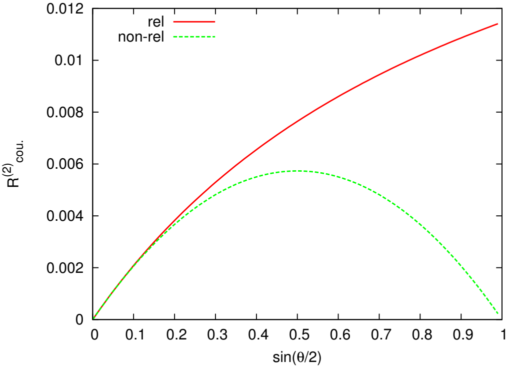
$$\frac{d\sigma^{(2)}_{cou}}{d\Omega} = \frac{1}{16\pi^2}\sum_{spin}\mathrm{Re}\left(\mathcal{M}^{(1)*}\mathcal{M}^{(2)}\right) \tag{120}$$

$$= \frac{1}{16\pi^2}\frac{Z^3e^6}{|\mathbf{q}|^2}\left[4\mathrm{ReI}_1E^3(1+\beta^2\cos\theta)+4E\mathrm{Re}\mathbf{I}_2\cdot(\mathbf{p}+\mathbf{p}')\right] \tag{121}$$

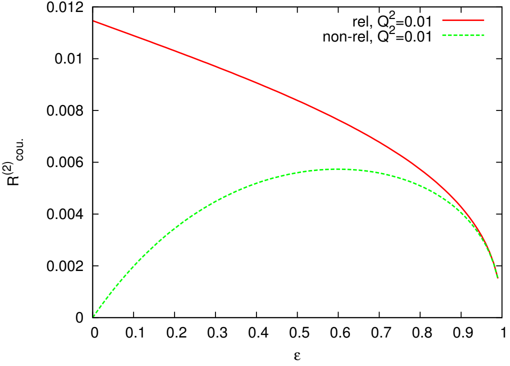
$$= 2\frac{Z^3\pi\alpha^3E}{|\mathbf{q}|^3}\left(1-\sin\frac{\theta}{2}\right) \tag{122}$$

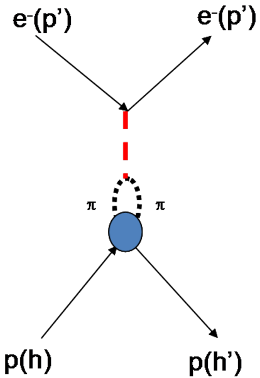
$$= \frac{Z^3\pi\alpha^3}{\mathbf{q}^2\beta}\left(1-\sin\frac{\theta}{2}\right) \tag{123}$$

$$R^{(2)}_{coul.} = \frac{\sigma^{(2)}_{coul.}}{\sigma_{1\gamma}} = \pi Z\alpha\beta\frac{\left(1-\sin\frac{\theta}{2}\right)}{\left(1-\beta^2\sin^2\frac{\theta}{2}\right)}\sin\frac{\theta}{2} \tag{124}$$

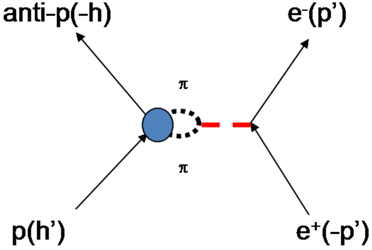


rel
non-rel
0.012
0.01
0.008
0.006
0.004
0.002
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$R^{(2)}_{cou.}$
sin(θ/2)


rel, Q2=0.01
non-rel, Q2=0.01
R(2)cou.
ε
0.012
0.01
0.008
0.006
0.004
0.002
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



$t=(p-p')^2=(h'-h)^2<0$



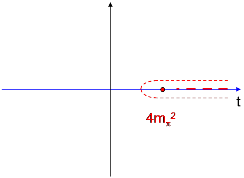
$t=(p+p')^2=(h'+h)^2>4\,m_\pi^2$

Interesting property:

$$\mathrm{Im}F_{1,2} = \frac{p_t^3}{\sqrt{t}}\Gamma^*_{1,2}(t) \tag{125}$$

p_t pion momentum in the crossed $(t-)$channel, $\Gamma_{1,2}$ – P-amplitudes for the $\pi\pi - N\overline{N}$.

Form-Factors: Time-like-Region



$$F(q^2) = \frac{1}{2\pi i}\oint_C dt\,\frac{F(t)}{t-q^2} = \frac{1}{2\pi i}\left(\int_{\infty}^{4m_\pi^2} dt\,\frac{F(t-i\epsilon)}{t-q^2-i\epsilon} + \int_{4m_\pi^2}^{\infty} dt\,\frac{F(t+i\epsilon)}{t-q^2+i\epsilon}\right) \quad (126)$$

$$= \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt\,\frac{\mathrm{Im}F(t+i\epsilon)}{t-q^2}, \quad F(s^*) = F^*(s) \quad (127)$$

$$= F(0) + \frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty} dt\,\frac{\mathrm{Im}F(t+i\epsilon)}{t(t-q^2)} \quad (128)$$

$$\langle r^2\rangle = \underbrace{\frac{6}{\pi}\int_{4m_\pi^2}^{\infty} dt\,\frac{\mathrm{Im}F_1^p(t)}{t^2}}_{\approx 0.65\ \mathrm{fm}^2} + \underbrace{\frac{F_2^p(0)}{4M^2}}_{\frac{3\kappa_p}{2M^2}\approx 0.12\ \mathrm{fm}^2} \quad (129)$$

The proton charge radius is governed by the low mass part of the spectrum of F_1^p!

Various Charge Distributions: Trick

Now we can apply the method proposed by R.R. Lewis, Jr. Phys. Rev. 102, (1956) 537.

$$F(\mathbf{K}) = \frac{1}{i\pi}\int_C ds \frac{sF(s)}{s^2 - \mathbf{K}^2 - i\epsilon}, \tag{130}$$

where the contour C is along real axis, avoiding singularities at $\pm K$. It is such strange representation since we have form factor depending on $|\mathbf{K}|^2$

The the second Born contribution reads

$$\begin{aligned} \mathcal{M}^{(2)} &= \frac{e^4}{\pi^2}\int dssF(s)\int dttF(t)\int \frac{d^3l}{(2\pi)^3} \\ &\times \frac{\overline{u}_{s'}(p')(\gamma^0E + \vec{\gamma}\cdot\mathbf{l} + m_e)u_s(p)}{(s^2-(\mathbf{p}'-\mathbf{l})^2)(t^2-(\mathbf{l}-\mathbf{p})^2)(\mathbf{l}^2-\mathbf{p}^2-i\epsilon)((\mathbf{p}'-\mathbf{l})^2+\mu^2)((\mathbf{l}-\mathbf{p})^2+\mu^2)} && (131) \\ &= \frac{e^4}{\pi^2}\int dssF(s)\int dttF(t)\int \frac{d^3l}{(2\pi)^3}\frac{\overline{u}_{s'}(p')(\gamma^0E + \vec{\gamma}\cdot\mathbf{l} + m_e)u_s(p)}{D} && (132) \\ D &= D_0D_1(s^2)D_2(t^2)D_1(0)D_2(0) && (133) \end{aligned}$$

where

$$\begin{aligned} D_0 &= \mathbf{l}^2 - \mathbf{p}^2 - i\epsilon && (134) \\ D_1(\lambda) &= (\mathbf{p}' - \mathbf{l})^2 - \lambda && (135) \\ D_2(\lambda) &= (\mathbf{p} - \mathbf{l})^2 - \lambda && (136) \\ & && (137) \end{aligned}$$

Denominator reads

$$D = D_0 D_1(s^2) D_2(t^2) D_1(-\mu^2) D_2(-\mu^2) \quad (138)$$

Notice useful property:

$$\frac{1}{D_1(\lambda_1)D_1(\lambda_2)} = \frac{1}{((\mathbf{p}'-\mathbf{l})^2-\lambda_1)((\mathbf{p}'-\mathbf{l})^2-\lambda_2)} = \frac{1}{\lambda_1-\lambda_2}\left[\frac{1}{((\mathbf{p}'-\mathbf{l})^2-\lambda_1)} - \frac{1}{((\mathbf{p}'-\mathbf{l})^2-\lambda_2)}\right] \quad (139)$$

$$= \frac{1}{\lambda_1-\lambda_2}\left[\frac{1}{D_1(\lambda_1)} - \frac{1}{D_1(\lambda_2)}\right] \quad (140)$$

Analogical partition can be done withe D_2 type of the denominator.
Finally we get the partition

$$\frac{1}{D} = \frac{1}{(s^2+\mu^2)(t^2+\mu^2)}\frac{1}{D_0}\left(\frac{1}{D_1(s^2)} - \frac{1}{D_1(-\mu^2)}\right)\left(\frac{1}{D_2(t^2)} - \frac{1}{D_2(-\mu^2)}\right) \quad (141)$$

$$= \frac{1}{(s^2+\mu^2)(t^2+\mu^2)}$$

$$\times \left(\frac{1}{D_0 D_1(s^2) D_2(t^2)} - \frac{1}{D_0 D_1(s^2) D_2(-\mu^2)} - \frac{1}{D_0 D_2(-\mu^2) D_1(t^2)} + \frac{1}{D_0 D_1(-\mu^2) D_2(-\mu^2)}\right) \quad (142)$$

We see that in order to evaluate the $\mathcal{M}^{(2)}$ we need to compute the two types of the integrals:

$$I_1(s^2,t^2) = \int \frac{d^3 l}{(2\pi)^3}\frac{1}{D} \tag{143}$$
$$I_2(s^2,t^2) = \int \frac{d^3 l}{(2\pi)^3}\frac{\mathbf{l}}{D} \tag{144}$$
$$\tag{145}$$

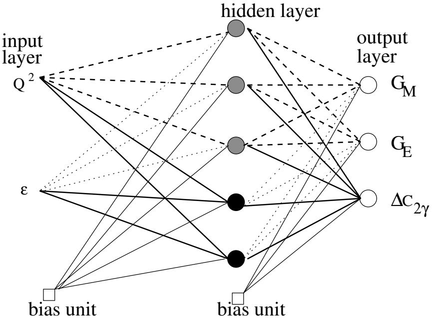
where

$$I_i = \frac{1}{(s^2+\mu^2)(t^2+\mu^2)}\Big(I_i(s^2,t^2) - I_i(s^2,-\mu^2) - I_i(-\mu^2,t^2) + I_i(-\mu^2,-\mu^2)\Big)\,, \tag{146}$$

where

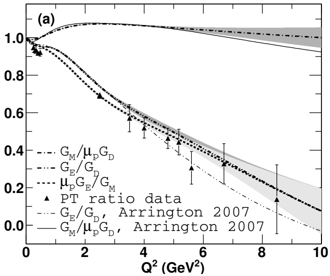
$$I_1(\lambda_1,\lambda_2) = \int \frac{d^3 l}{(2\pi)^3}\frac{1}{D_0 D_1(\lambda_1)D(\lambda_2)} \tag{147}$$
$$I_2(\lambda_1,\lambda_2) = \int \frac{d^3 l}{(2\pi)^3}\frac{\mathbf{l}}{D_0 D_1(\lambda_1)D(\lambda_2)} \tag{148}$$
$$\tag{149}$$

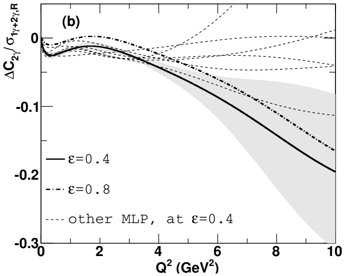
Neural Network Attempt



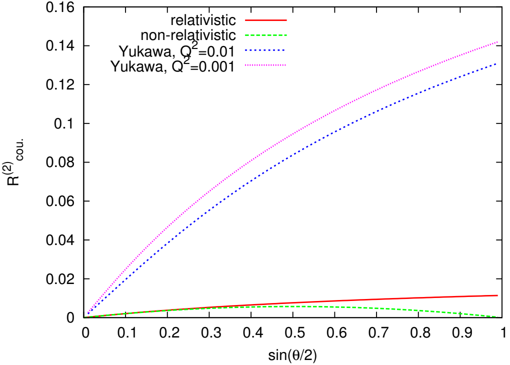
$$\sigma_{1\gamma+2\gamma,R}(Q^2,\epsilon) = \sigma_{1\gamma,R}(Q^2,\epsilon)+\Delta C_{2\gamma}(Q^2,\epsilon), \quad \mathcal{R}_{1\gamma}(Q^2) = \mu_p \frac{G_E(Q^2)}{G_M(Q^2)}, \quad \mathcal{R}_{\pm}(Q^2,\epsilon) = 1 - \frac{2\Delta C_{2\gamma}(Q^2,\epsilon)}{\sigma_{1\gamma+2\gamma,R}(Q^2,\epsilon)}. \tag{150}$$

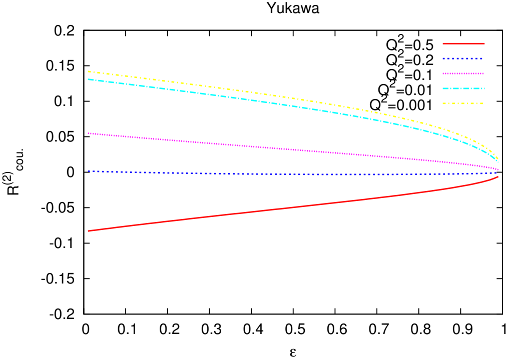
K.M. Graczyk, Phys. Rev. C 84, 034314 (2011).

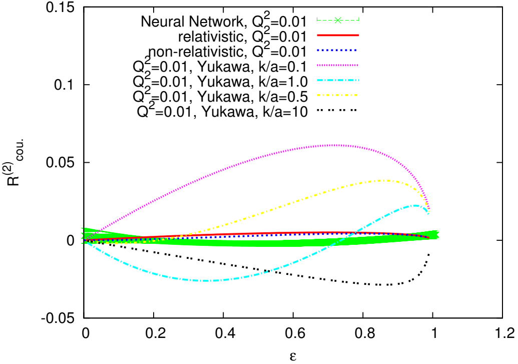




from K.M. Graczyk, Phys. Rev. C 84, 034314 (2011)


relativistic
non-relativistic
Yukawa, Q^2=0.01
Yukawa, Q^2=0.001
R^(2)_cou.
sin(θ/2)
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Yukawa
Q^2=0.5
Q^2=0.2
Q^2=0.1
Q^2=0.01
Q^2=0.001
R^(2)_cou.
ε
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Neural Network, Q2=0.01
relativistic, Q2=0.01
non-relativistic, Q2=0.01
Q2=0.01, Yukawa, k/a=0.1
Q2=0.01, Yukawa, k/a=1.0
Q2=0.01, Yukawa, k/a=0.5
Q2=0.01, Yukawa, k/a=10
R(2)cou.
ε
0.15
0.1
0.05
0
-0.05
0
0.2
0.4
0.6
0.8
1
1.2

BACKUP SLIDES

It can be easily shown that with the help of the Gauss theorem one immediately gets:

$$\int_V d^3x \triangle \left(\frac{1}{|\mathbf{x}|}\right) = \int_V d^3x \nabla \cdot \left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \quad (151)$$

$$\underbrace{=}_{Gauss\ Theorem} \int_{\partial V} da \left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \cdot \mathbf{n} \quad (152)$$

$$= -\lim_{r\to\infty} 4\pi|\mathbf{x}|^2 \frac{1}{|\mathbf{x}|^2} = -4\pi\tilde{\delta} \quad (153)$$

Therefore

$$\triangle \left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\delta(\mathbf{x}). \quad (154)$$

Notice that the Fourier transform of the δ denoted as $\tilde{\delta}$ reads as

$$\tilde{\delta}(k) = \int d^3x \delta(x) e^{-i\mathbf{k}\cdot\mathbf{x}} = 1. \quad (155)$$