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(1) Isospin 1 and 1/2 generators $t_j$ $\tau_j$

$[t_j, t_k] = i\epsilon_{jkl} t_l$

$[\tau_j, \tau_k] = 2i\epsilon_{jkl}\tau_l$ $\qquad \tau_j\tau_k = \delta_{jk} + i\epsilon_{jkl}\tau_l$

$\vec{t}^{\,2} = 2$

$\vec{\tau}^{\,2} = \cancel{\tfrac{3}{4}}\; 3$ (little misleading!)

(2) $(\vec{t}\vec{\tau})^2 = 2 - \vec{t}\vec{\tau}$ $\qquad$ here $\tau_j$ are Pauli matrices!

$x^2 + x - 2 = 0 \Rightarrow x_\pm = 1, -2$

Two projection operator $\underbrace{\frac{1}{3}(2 + \vec{t}\vec{\tau})}_{P_{3/2}}$ $\quad \underbrace{\frac{1}{3}(1 - \vec{t}\vec{\tau})}_{P_{1/2}}$

$P_{1/2}P_{3/2} = P_{3/2}P_{1/2} = 0$

$P_{3/2}^2 \qquad P_{1/2}^2$

(3) Physical interpretation

$\langle \pi_b | \vec{t}\vec{\tau} | \pi_a \rangle = i\epsilon_{bca}\tau_c = \frac{1}{2}[\tau_a, \tau_b]$

$= \frac{1}{2}(\tau_a\tau_b \mp \tau_b\tau_a \pm \tau_b\tau_a) = \delta_{ab} - \tau_b\tau_a$

Special case $\langle \pi_+ | \vec{t}\vec{\tau} | \pi_+ \rangle = ?$

A rule is $|\pi_+\rangle = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)$

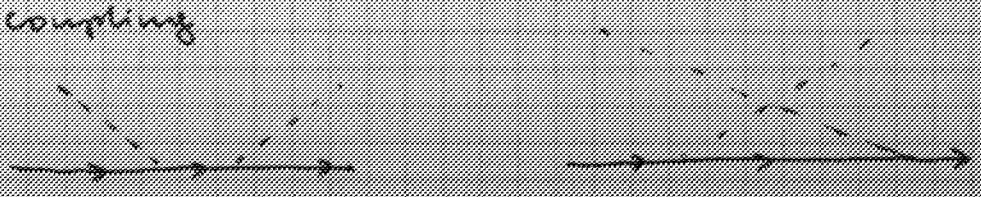
$\langle \pi_+ | = \frac{1}{\sqrt{2}}\langle \pi_1 - i\pi_2 |$

What is $\delta_{++}$ ? $\quad \delta_{-+} = 1$

$\langle \pi_+ | \vec{t}\vec{\tau} | \pi_+ \rangle = 1 - \tau_-\tau_+ = 1 - \frac{2}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

We get $\langle p\pi^+ | \vec{\tau}\vec{\tau} | p\pi^+ \rangle = 1$ as expected

(4) How is it related to amplitudes etc?
Consider $\pi N$ scattering with pseudoscalar coupling



~~Pions in the initial and final states are described by $\Phi$ and $\Phi^*$, where~~

~~$\Phi_{\pi^\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix}$, $\Phi_{\pi^0} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$~~

to be skipped

The amplitude is

$$\mathcal{M} = \chi'^{\dagger}\, \bar{u}(\vec{p}\,',s')\, g_0^2 \left\{ \vec{\tau}\vec{\Phi}_b^{*}\, \gamma_5 \frac{i}{\quad} \gamma_5 \vec{\tau}\vec{\Phi}_a + \vec{\tau}\vec{\Phi}_a\, \gamma_5 \frac{i}{\quad} \gamma_5 \vec{\tau}\vec{\Phi}_b^{*} \right\} u(\vec{p},s)\,\chi$$

The general form is

$$\mathcal{M} = \chi'^{\dagger}\, \bar{u}(\vec{p}\,',s')\, T_{ba}\, u(\vec{p},s)\, \chi$$

$$T_{ba} = g_0^2 \left\{ \tau_b \tau_a \frac{i(M - \not{p} - \not{q})}{(p+q)^2 - M^2} + \tau_a \tau_b \frac{i(\quad)}{\quad} \right\}$$

relevant isospin structure

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## ⑤ Amplitude decomposition

$$T_{ba} = \tau_b \tau_a \, (NP) + \tau_a \tau_b \, (CNP)$$

We use $\quad \langle \pi_b | \bar{t}\bar{\tau} | \pi_a \rangle = \delta_{ab} - \tau_b \tau_a$

$$T_{ba} = \cancel{\delta_{ab}} \, T^{3/2} \, \frac{2 + \bar{t}\bar{\tau}}{3} + T^{1/2} \, \frac{1 - \bar{t}\bar{\tau}}{3}$$

$$T_{ba} = \left( T^{3/2} \frac{2}{3} + T^{1/2} \frac{1}{3} \right) \delta_{ba} + \left( T^{3/2} \frac{1}{3} - T^{1/2} \frac{1}{3} \right) \left( \delta_{ab} - \tau_b \tau_a \right)$$

$$= T^{3/2} \delta_{ba} + \frac{1}{3} \left( T^{1/2} - T^{3/2} \right) \tau_b \tau_a$$

We would like to have $\quad \tau_b \tau_a \, ,\dots \quad \tau_a \tau_b \, \dots$

$$\delta_{ba} = \frac{1}{2} \left( \tau_b \tau_a + \tau_a \tau_b \right)$$

$$T_{ba} = \frac{1}{2} T^{3/2} \tau_a \tau_b + \tau_b \tau_a \left( \frac{1}{3} T^{1/2} + \frac{1}{6} T^{3/2} \right)$$

By comparing $\quad NP = \frac{1}{3} T^{1/2} + \frac{1}{6} T^{3/2}$

$$CNP = \frac{1}{2} T^{3/2}$$

$$\boxed{\begin{aligned} T^{3/2} &= 2 (CNP) \\ T^{1/2} &= 3 (NP) - (CNP) \end{aligned}}$$

(with only NP the process $p\pi^+ \to p\pi^+$ is impossible)

## ⑥ Towards Adler

Adler defines $T^{\pm}$ amplitudes in the following way

$$T_{ba} = T^+ \frac{1}{2}\{\tau_b,\tau_a\} + T^- \frac{1}{2}[\tau_b,\tau_a]$$

$$= \delta_{ab} T^+ + \frac{1}{2}[\tau_b,\tau_a] T^-$$

One can easily ~~express~~ find relations

$$T^\pm \Leftrightarrow T^{1/2}, T^{3/2}$$

$$T_{ba} = T^{3/2}\delta_{ba} + \frac{1}{3}\left(T^{1/2}-T^{3/2}\right)\tau_b\tau_a$$

$$[\tau_b,\tau_a] = \tau_b\tau_a - \tau_a\tau_b$$

$$\tau_a\tau_b + \tau_b\tau_a = 2\delta_{ba}$$

$$[\tau_b,\tau_a] = \tau_b\tau_a - \left(2\delta_{ba} - \tau_b\tau_a\right) = 2\tau_b\tau_a - 2\delta_{ba}$$

$$T_{ba} = \delta_{ab} T^+ + (\tau_b\tau_a - \delta_{ba})T^- = \delta_{ab}(T^+ - T^-) + \tau_{ba} T^-$$

$$\boxed{\begin{aligned} T^+ - T^- &= T^{3/2} \\ T^- &= \frac{1}{3}\left(T^{1/2} - T^{3/2}\right)\end{aligned}}$$

(7) Consider now weak$^{cc}$ pion production.

Instead of $\langle \pi N | S-1 | \pi N\rangle$ we have

$$\langle \pi N | S-1 | N W^-\rangle$$

From the isospin point of view $W^-$ is like $|1,1\rangle$

$p \to p\pi^+ \qquad n \to n\pi^+ \qquad n \to p\pi^0$

i.e. $W^-$ is like $\pi^+$ !

Thus, we know that the general amplitude must be of the form

$$\boxed{T_{b+} = T^{3/2}\,\delta_{b+} + \frac{1}{3}\left(T^{1/2} - T^{3/2}\right)\tau_b\tau_+}$$

⑧ There are three CC $\nu$ channels

$p \to p\pi^+$

$n \to n\pi^+$

$n \to p\pi^0$

Let us analyze them one by one.

$\pi^+$ in the final state corresponds to $b = -$

$T_{b+}$ becomes $\quad T^{3/2} + \frac{1}{3}\left(T^{1/2} - T^{3/2}\right)\tau_-\tau_+$

$$\tau_+ = \sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$\tau_- = \sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\tau_-\tau_+ = 2\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$p \to p\pi^+ \quad \Rightarrow \quad T^{3/2}$

$n \to n\pi^+ \quad \Rightarrow \quad T^{3/2} + \frac{2}{3}\left(T^{1/2} - T^{3/2}\right) = \frac{2}{3}T^{1/2} + \frac{1}{3}T^{3/2}$

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Finally $n \to p\pi^0$

$$T_{3+} = T^{3/2}\delta_{3+} + \frac{1}{3}\left(T^{1/2} - T^{3/2}\right)\tau_3\tau_+$$

$$\tau_3\tau_+ = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\sqrt{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \sqrt{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\langle p | \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} | n\rangle = 1$$

$$n \to p\pi^0 \quad \Rightarrow \quad \frac{\sqrt{2}}{3}\left(T^{1/2} - T^{3/2}\right)$$

(9) If there is only $T^{3/2}$ amplitude, there is a simple relation between cross sections

$$\sigma(p\pi^+) : \sigma(n\pi^+) : \sigma(p\pi^0) = 1 : \frac{1}{9} : \frac{2}{9}$$

$$\sigma(p\pi) : \sigma(n\pi) = 1 : \frac{1}{3}$$

(10) Neutron channels „measure" $T^{1/2}$ and interference between $T^{1/2}$ and $T^{3/2}$

Suppose $\quad T^{3/2} = |T^{3/2}|\, e^{i\alpha}$

$\quad T^{1/2} = |T^{1/2}|\, e^{i\alpha}\, e^{i\Phi}$ ← relative phase

Then

$$\sigma(p\pi^+) \sim |T^{3/2}|^2$$

$$\sigma(p\pi^0) \sim \frac{2}{9}\left(|T^{1/2}|^2 + |T^{3/2}|^2 - 2|T^{3/2}||T^{1/2}|\cos\Phi\right)$$

$$\sigma(n\pi^+) \sim \frac{4}{9}|T^{1/2}|^2 + \frac{1}{9}|T^{3/2}|^2 + \frac{4}{9}|T^{1/2}||T^{3/2}|\cos\Phi$$

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(11) The simplest estimate of $\frac{|T^{1/2}|}{|T^{3/2}|}$ and $\Phi$ is done by three cross section measurements:

$$\sigma(p\pi^0) + \sigma(n\pi^+) \sim \frac{2}{3}|T^{1/2}|^2 + \frac{1}{3}|T^{3/2}|^2$$

$$\boxed{\frac{\sigma(n\pi)}{\sigma(p\pi)} = \frac{1}{3} + \frac{2}{3}\left(\frac{|T^{1/2}|}{|T^{3/2}|}\right)^2}$$

$$\sigma(n\pi^+) - 2\sigma(p\pi^0) \sim -\frac{1}{3}|T^{3/2}|^2 + \frac{4}{3}|T^{3/2}||T^{1/2}|\cos\Phi$$

$$\boxed{\frac{\sigma(n\pi^+) - 2\sigma(p\pi^0)}{\sigma(p\pi^+)} = -\frac{1}{3} + \frac{4}{3}\frac{|T^{1/2}|}{|T^{3/2}|}\cdot\cos\Phi}$$

(12) ANL experiment simply counted numbers of events ~~tuch~~ three channels (cut $W<1.4$ GeV), see Table VII

$p\pi^+ \to 716.5$

$p\pi^0 \to 236.2$

$n\pi^+ \to 225.6$

$$\frac{1}{3} + \frac{2}{3}\left(\frac{T^{1/2}}{T^{3/2}}\right)^2 = \frac{461.8}{716.5} \quad \Rightarrow \quad \frac{T^{1/2}}{T^{3/2}} \simeq \sqrt{0.467} \sim 0.68$$

$$\frac{246.8}{716.5} = \frac{1}{3} - \frac{4}{3}\frac{T^{1/2}}{T^{3/2}}\cos\Phi \quad \Rightarrow \quad \cos\Phi \simeq -0.012$$

$$\cos\Phi \simeq 90.7^\circ$$

~~Opposite phase would~~

This means that changing the relative sign has very little effect on cross sections!

(13) Another example is F-N model. The cross section table allows for computation of $\left|\frac{T^{1/2}}{T^{3/2}}\right|$ and $\cos\Phi$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $p\pi^+$ | 0.117 | 0.287 | 0.408 | 0.476 | 0.508 | 0.522 | 0.527 |
| $n\pi^+$ | 0.018 | 0.058 | 0.099 | 0.130 | 0.149 | 0.161 | 0.167 |
| $p\pi^0$ | 0.039 | 0.095 | 0.140 | 0.169 | 0.188 | 0.200 | 0.206 |

ANL $p\pi^+$ 0.231
$p\pi^0$ 0.079
$n\pi^+$ 0.052

$R = 0.59$

$\Phi = 99.2$